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LOCALIZED WAVE FORMS OF MOTION OF AN INFINITE SHELL OF REVOLUTION†

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The wave forms of motion of a thin infinite shell of revolution defined by initial displacements and velocities localized in the neighbourhood of a parallel are studied. Using the method developed in [1], a solution of the equations of motion is constructed as a superposition of tangential and flexural wave packets propagating in the direction of the shell axis. As an example, an elongated conical shell is considered. © 1997 Elsevier Science Ltd. All rights reserved.

The problem of the propagation of localized families of flexural waves in a cylindrical shell of moderate length was considered in [3] using Maslov's method [2]. The behaviour of the wave packets was found to depend on the geometry of the shell, and it was observed, in particular, that it is possible for the waves to be reflected from generatrices on which the surface of the shell is extremely curved.

1. FORMULATION OF THE PROBLEM

Consider an infinite shell of revolution of thickness h. Let $s = Rx$ be the arc length of the generatrix ($-\infty < x < +\infty$) and let φ be the angle measured around the shell. Here $R = B(0)$ is the characteristic dimension of the shell, where $B(x) = Rf(x)$ is the distance to the axis of revolution. In this system of coordinates the first quadratic form of the surface has the form $R^2(dx^2 + f^2 d\varphi^2)$. The principal radii of curvature R_1 and R_2 satisfy the relations

$$R_1 = -R\frac{\sqrt{1-f'^2}}{f''}, \quad R_2 = R\frac{f}{\sqrt{1-f'^2}}$$

Here and henceforth we use a prime to denote the derivative with respect to x.

We shall study non-axially-symmetric forms of motion with wave number m_0 around the shell. We begin with the linear equations

$$(\mathbf{L} + \partial^2/\partial t^2)\mathbf{U}^T = 0 \tag{1.1}$$

based on the Kirchoff–Love hypotheses. Here $\mathbf{U} = (u_1, u_2, u_3)$, $u_3^* = R\cos(m_0\varphi)u_3$ is the normal, and $u_1^* = R\cos(m_0\varphi)u_1$ and $u_2^* = R\sin(m_0\varphi)u_2$ are the tangential displacements of the points on the median surface in the axial and circumferential directions, respectively, $t = t^*/T^*$ is the dimensionless time, $T^* = R[(1-\nu^2)\rho/E]^{1/2}$ is the characteristic time, and ρ, ν and E are the density of the material, Poisson's ratio and Young's modulus, respectively. We will denote by the 3×3 matrix formed by the operators [4, p. 104]

$$L_{11}z = -\frac{\partial}{\partial x}\frac{1}{f}\frac{\partial}{\partial x}(fz) + (1-\nu)\left(\frac{m_0^2}{2f^2} - k_1k_2\right)z$$

$$L_{12}z = -m_0\frac{\partial}{\partial x}\left(\frac{z}{f}\right) + \frac{(1-\nu)m_0}{2f^2}\frac{\partial}{\partial x}(fz)$$

$$L_{13}z = \frac{\partial}{\partial x}\left[(k_1+k_2)z\right] - (1-\nu)k_2\frac{\partial z}{\partial x}$$

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$$L_{21}z = \frac{m_0}{f^2}\frac{\partial}{\partial x}(fz) - \frac{(1-\nu)m_0}{2}\frac{\partial}{\partial x}\left(\frac{z}{f}\right)$$

$$L_{22}z = \frac{1-\nu}{2}\frac{\partial}{\partial x}\frac{1}{f}\frac{\partial}{\partial x}(fz) + \left[\frac{m_0}{f^2} - (1-\nu)k_1k_2\right]z$$

$$L_{23}z = -\frac{m_0}{f}(k_2+\nu k_1)z, \quad L_{31}z = -\frac{1}{f}(k_1+k_2)\frac{\partial}{\partial x}(fz) + \frac{1-\nu}{f}\frac{\partial}{\partial x}(k_2 fz), \quad L_{32} = L_{23}$$

$$L_{33}z = \varepsilon^4\left\{\left[\frac{1}{f}\left(\frac{\partial}{\partial x}f\frac{\partial}{\partial x} - \frac{m_0^2}{f}\right)\right]^2 z - \frac{1}{(1+\nu)f}\left(\frac{\partial}{\partial x}f''\frac{\partial}{\partial x}z\right) + \frac{m_0^2 f''}{f^2}z\right\} +$$

$$+\left(k_1^2 + 2\nu k_1k_2 + k_2^2\right)z$$

where $\varepsilon^4 = h^2/(12R^2)$ is a small parameter, $k_1 = R/R_1$, and $k_2 = R/R_2$.

Suppose that an initial wave packet with centre on the parallel $x = 0$ is given

$$u_j\big|_{t=0} = \lambda_j^0(\zeta,\varepsilon)\exp\{i\varepsilon^{-1}S_0(x)\}, \quad j = 1,2,3$$
$$\dot{u}_j\big|_{t=0} = i\varepsilon_j\eta_j^0(\zeta,\varepsilon)\exp\{i\varepsilon^{-1}S_0(x)\} \tag{1.2}$$

where

$$S_0(x) = a_0x + \tfrac{1}{2}b_0x^2, \quad \operatorname{Im} b_0 > 0$$
$$\lambda_j^0 = \sum_{k=0}^{\infty}\varepsilon^{k/2}\lambda_{jk}(\zeta), \quad \eta_j^0 = \sum_{k=0}^{\infty}\varepsilon^{k/2}\eta_{jk}(\zeta) \tag{1.3}$$
$$\zeta = \varepsilon^{-1/2}x, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon^{-1}, \quad \varepsilon_3 = 1$$

Here $\lambda_{jk}(\zeta)$ and $\eta_{jk}(\zeta)$ are polynomials of degree M_{jk} and K_{jk}, respectively, with complex coefficients. Differentiation with respect to time t is denoted by a dot.

Here we shall consider the case when $m_0 = \varepsilon^{-1}m$, where $m \sim 1$.

2. TANGENTIAL WAVES

From (1.2) it follows that $\partial/\partial x \sim \varepsilon^{-1}$ in (1.1). Let $\partial/\partial t \sim \varepsilon^{-\alpha}$, where $\alpha \geqslant 0$. To study the tangential motion we set $u_1 = u_\tau$, $u_2 = v_\tau$, $u_3 = \varepsilon^{\beta}w_\tau$, where $\beta \geqslant 0$ and $u_\tau, v_\tau, w_\tau \sim 1$. It is required that the terms containing the leading derivatives with respect to x in the first and second expressions in (1.1), and the inertial terms should be infinitesimals of the same order and be contained in the asymptotically leading part of the resolving system, ensuring that the integrals are of wave type. Hence we find that $\alpha = \beta = 1$ and the equations for the tangential waves have the form

$$\left(\mathbf{L} + \partial^2/\partial t^2\right)\left(\mathbf{E}_{\tau\varepsilon}\mathbf{U}_\tau^T\right) = 0$$
$$\mathbf{E}_{\tau\varepsilon} = \operatorname{diag}(1,1,\varepsilon), \quad \mathbf{U}_\tau = (u_\tau, v_\tau, w_\tau) \tag{2.1}$$

Following [1], we shall seek a solution of (2.1) with initial conditions (1.2) as a wave packet with centre at $x = q_\tau(t)$, where $q_\tau(t)$ is a twice differentiable function such that $q_\tau(0) = 0$. In (2.1) we change to a new system of coordinates using the formula

$$x = q_\tau(t) + \varepsilon^{1/2}\xi_\tau$$

We expand each of the functions in (2.1) in a Taylor series in powers of $\varepsilon^{1/2}\xi_\tau$ in the neighbourhood of $q_\tau(t)$. Let Q_τ be the domain of variation of $q_\tau(t)$. We assume that for any $q_\tau \in Q_\tau$ the values of all the coefficients and their derivatives in (2.1) are of order one as $\varepsilon \to 0$.

We will seek an asymptotic solution of (2.1) of the form

$$\mathbf{U}_\tau = \sum_{k=0}^{\infty} \varepsilon^{k/2}\mathbf{U}_{\tau,k} \exp\{i\varepsilon^{-1}S_\tau(\xi_\tau, t, \varepsilon)\}$$
$$S_\tau = \int_0^t \omega_\tau(t')dt' + \varepsilon^{1/2}p_\tau(t)\xi_\tau + \frac{1}{2}\varepsilon b_\tau(t)\xi_\tau^2 \tag{2.2}$$
$$\mathbf{U}_{\tau,k} = (u_{\tau,k}, v_{\tau,k}, w_{\tau,k})$$

where $u_{\tau,k}(\xi_\tau, t)$, $v_{\tau,k}(\xi_\tau, t)$, $w_{\tau,k}(\xi_\tau, t)$ are polynomials in ξ_τ, and $\operatorname{Im} b_\tau(t) > 0$ for any $t \geqslant 0$. To determine the unknown functions in (2.2) we substitute (2.2) into (2.1) and equate the coefficients of like powers of $\varepsilon^{1/2}$. As a result, we obtain a sequence of equations

$$\sum_{j=0}^{k} \mathbf{L}_{\tau,j}\mathbf{U}_{\tau,k-j}^T = 0, \quad k = 0,1,2,\ldots \tag{2.3}$$

Here $\mathbf{L}_{\tau,0}$ is a 3×3 matrix with elements

$$l_{\tau,11} = p_\tau^2 + \frac{(1-\nu)m^2}{2f^2} - (\omega_\tau - \dot{q}_\tau p_\tau)^2, \quad l_{\tau,12} = -\frac{im(1+\nu)p_\tau}{2f}$$

$$l_{\tau,21} = -l_{\tau,12}, \quad l_{\tau,22} = \frac{1-\nu}{2}p_\tau^2 + \frac{m^2}{f^2} - (\omega_\tau - \dot{q}_\tau p_\tau)^2$$

$$l_{\tau,31} = -ip_\tau(k_1 + \nu k_2), \quad l_{\tau,32} = -\frac{m}{f}(k_2 + \nu k_1)$$

$$l_{\tau,33} = -(\omega_\tau - \dot{q}_\tau p_\tau)^2, \quad l_{\tau,13} = l_{\tau,23} = 0$$

and $\mathbf{L}_{\tau,j}$ ($j \geqslant 1$) are matrix-valued operators defined by

$$\mathbf{L}_{\tau,1} = \left(b_\tau \frac{\partial \mathbf{L}_{\tau,0}}{\partial p_\tau} + \frac{\partial \mathbf{L}_{\tau,0}}{\partial q_\tau} + \dot{p}_\tau \frac{\partial \mathbf{L}_{\tau,0}}{\partial \omega_\tau}\right)\xi_\tau - i\frac{\partial \mathbf{L}_{\tau,0}}{\partial p_\tau}\frac{\partial}{\partial \xi_\tau}$$

$$\mathbf{L}_{\tau,2} = \frac{1}{2}\left(b_\tau^2 \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau^2} + 2b_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau \partial q_\tau} + \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial q_\tau^2} + \dot{p}_\tau^2 \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau^2} + 2\dot{p}_\tau b_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau \partial p_\tau} + \dot{b}_\tau \frac{\partial \mathbf{L}_{\tau,0}}{\partial \omega_\tau}\right)\xi_\tau^2 -$$
$$-\frac{1}{2}\frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau^2}\frac{\partial^2}{\partial \xi_\tau^2} - i\left(b_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau^2} + \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau \partial q_\tau} + \dot{p}_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau \partial p_\tau}\right)\xi_\tau \frac{\partial}{\partial \xi_\tau} - i\frac{\partial \mathbf{L}_{\tau,0}}{\partial \omega_\tau}\frac{\partial}{\partial t} - \tag{2.4}$$
$$-i\left(\frac{1}{2}b_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial p_\tau^2} + \frac{1}{2}\dot{\omega}_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau^2} + \dot{p}_\tau \frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau \partial p_\tau} + \mathbf{N}_\tau\right), \ldots$$

In (2.4) $\mathbf{N}_\tau$ denotes a matrix with elements

$$n_{\tau,11} = \frac{f}{f'}p_\tau + \ddot{q}_\tau p_\tau, \quad n_{\tau,12} = \frac{i(3-\nu)mf'}{2f^2}, \quad n_{\tau,13} = 0$$

$$n_{\tau,21} = n_{\tau,12}, \quad n_{\tau,22} = \frac{(1-\nu)f'p_\tau}{2f} + \ddot{q}_\tau p_\tau, \quad n_{\tau,23} = 0$$

$$n_{\tau,31} = i\left[\frac{(1-\nu)(k_2 f)'}{f} - \frac{(k_1 + k_2)f'}{f}\right], \quad n_{\tau,32} = 0, \quad n_{\tau,33} = \ddot{q}_\tau p_\tau$$

Considering (2.3) for $k = 0$, we can find a relation between the instantaneous frequency $\omega_\tau(t)$ and the functions $p_\tau(t)$, $q_\tau(t)$

$$\omega_\tau^{(l)}(t)=\dot q_\tau^{(l)}(t)p_\tau^{(l)}(t)-H_\tau^{(l)}\left[p_\tau^{(l)}(t),q_\tau^{(l)}(t)\right],\quad l=1,2,3,4 \tag{2.5}$$

where

$$H_\tau^{(l)}(p,q)=\delta_l\left[p^2+\frac{m^2}{f^2(q)}\right]^{1/2},\quad \delta_1=1,\quad \delta_2=-1$$

$$\delta_3=\left(\frac{1-\nu}{2}\right)^{1/2},\quad \delta_4=-\delta_3$$

Here $H_\tau^{(1)}$, $H_\tau^{(2)}$ are the Hamilton functions corresponding to the two branches of longitudinal waves, while $H_\tau^{(3)}$, $H_\tau^{(4)}$ correspond to the two branches of torsional waves. Henceforth the superscript l will be omitted. It will be understood that packets of longitudinal or torsional waves are considered.

From (2.3) for $k = 0$ it follows that

$$\mathbf{U}_{\tau,0}=P_{\tau,0}(\xi_\tau,t)\mathbf{Y}_{\tau,0} \tag{2.6}$$

where

$$\mathbf{Y}_{\tau,0}=(y_1,y_2,y_3),\quad y_1=1,\quad y_2=-l_{\tau,11}l_{\tau,12}^{-1}$$
$$y_3=(l_{\tau,12}l_{\tau,33})^{-1}(l_{\tau,32}l_{\tau,11}-l_{\tau,31}l_{\tau,12}) \tag{2.7}$$

For $k = 1$ in (2.3) we have a non-homogeneous system of algebraic equations, which can be solved under the condition

$$\mathbf{Z}_{\tau,0}^{*}\mathbf{L}_{\tau,1}\mathbf{U}_{\tau,0}^{T}=0 \tag{2.8}$$

where $\mathbf{Z}_{\tau,0}$ is any non-trivial solution of the system adjoint to (2.3) for $k = 0$. By (2.6), condition (2.8) is a differential equation for $P_{\tau,0}(\xi_\tau, t)$. For this equation to have a solution that is a polynomial in ξ_τ it is necessary that $p_\tau(t)$ and $q_\tau(t)$ should be a solution of the Hamilton system

$$\dot q_\tau=\frac{\partial H_\tau}{\partial p_\tau},\quad \dot p_\tau=-\frac{\partial H_\tau}{\partial q_\tau} \tag{2.9}$$

Then

$$\mathbf{U}_{\tau,1}=P_{\tau,1}\mathbf{Y}_{\tau,0}+\xi_\tau P_{\tau,0}\left(b_\tau\frac{\partial \mathbf{Y}_{\tau,0}}{\partial p}+\frac{\partial \mathbf{Y}_{\tau,0}}{\partial q}\right)-i\frac{\partial P_{\tau,0}}{\partial \xi_\tau}\frac{\partial \mathbf{Y}_{\tau,0}}{\partial p_\tau} \tag{2.10}$$

where $P_{\tau,1}(\xi_\tau, t)$ is a polynomial in ξ_τ to be determined.

When (2.6), (2.9) and (2.10) are taken into account, the condition for (2.3) to be solvable for $k = 2$ gives again a differential equation for $P_{\tau,0}$

$$\mathbf{Z}_{\tau,0}^{*}\left(\mathbf{L}_{\tau,1}\mathbf{U}_{\tau,1}^{T}+\mathbf{L}_{\tau,0}\mathbf{U}_{\tau,0}^{T}\right)=0 \tag{2.11}$$

The latter has a polynomial solution if and only if $b_\tau(t)$ is a solution of the Riccati equation

$$\dot b_\tau+\frac{\partial^2 H_\tau}{\partial p_\tau^2}b_\tau^2+2\frac{\partial^2 H_\tau}{\partial p_\tau\partial q_\tau}b_\tau+\frac{\partial^2 H_\tau}{\partial q_\tau^2}=0 \tag{2.12}$$

Then (2.11) takes the form

$$\frac{1}{2}\frac{\partial^2 H_\tau}{\partial p_\tau^2}\frac{\partial^2 P_{\tau,0}}{\partial \xi_\tau^2}+i\left(b_\tau\frac{\partial^2 H_\tau}{\partial p_\tau^2}+\frac{\partial^2 H_\tau}{\partial p_\tau\partial q_\tau}\right)\xi_\tau\frac{\partial P_{\tau,0}}{\partial \xi_\tau}+i\frac{\partial P_{\tau,0}}{\partial t}+$$

$$+i\left(\mathbf{Z}^{*}_{\tau,0}\frac{\partial \mathbf{L}_{\tau,0}}{\partial \omega_\tau}\mathbf{Y}^{T}_{\tau,0}\right)^{-1}\mathbf{Z}^{*}_{\tau,0}\left[\left(\frac{1}{2}b_\tau\frac{\partial^2 H_\tau}{\partial p_\tau^2}\frac{\partial \mathbf{L}_{\tau,0}}{\partial \omega_\tau}+\frac{1}{2}\dot{\omega}_\tau\frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau^2}+\right.\right.$$

$$\left.\left.+\dot{p}_\tau\frac{\partial^2 \mathbf{L}_{\tau,0}}{\partial \omega_\tau \partial p_\tau}+\mathbf{N}_\tau\right)\mathbf{Y}^{T}_{\tau,0}+\frac{\partial \mathbf{L}_{\tau,0}}{\partial p_\tau}\frac{\partial \mathbf{Y}^{T}_{\tau,0}}{\partial q_\tau}\right]P_{\tau,0}=0 \tag{2.13}$$

Equation (2.13) has a polynomial solution

$$P_{\tau,0}=\sum_{k=0}^{\sigma_{\tau,0}}A_{\tau,0k}(t;c_{0i})\xi_\tau^k \tag{2.14}$$

of degree $\sigma_{\tau,0}$ whose coefficients $A_{\tau,0k}$ contain arbitrary constants c_{0i} ($i = 0, 1, \ldots, \sigma_{\tau,0}$). We will not give the explicit form of $A_{\tau,0k}$ because of its complexity.

When $k \geqslant 3$ the condition for (2.3) to be solvable leads to non-homogeneous differential equations for $P_{\tau,k-2}$.

3. TRANSVERSE WAVES

Let $\partial/\partial t \sim \varepsilon^{-\alpha}$, $u_1 = \varepsilon^{\beta}u_n$, $u_2 = \varepsilon^{\gamma}v_n$ and $u_3 = w_n$, where $\alpha, \beta, \gamma \geqslant 0$. We require that the inertial term and the terms containing higher derivatives in x in the third equation in (1.1) should be infinitesimals of the same order and appear in the leading asymptotic part of the resolving system. We find that $\alpha = 0$ and $\beta = \gamma = 1$. We introduce the "slow time" $t_1 = \varepsilon t$. We then obtain the system of equations

$$\left(\mathbf{L}+\varepsilon^2\partial^2/\partial t_1^2\right)\left(\mathbf{E}_{n\varepsilon}\mathbf{U}_n^T\right)=0$$
$$\mathbf{E}_{n\varepsilon}=\operatorname{diag}(\varepsilon,\varepsilon,1),\quad \mathbf{U}_n=(u_n,v_n,w_n) \tag{3.1}$$

for flexural waves.

As before, we seek a solution of (3.1) as a wave packet (2.2) with τ replaced by n and t replaced by t_1 in all formulae. Denoting the set of values of $q_n(t_1)$ by Q_n, we assume that for any $q_n \in Q_n$ all the coefficients and their derivatives in (3.1) are of order one.

The procedure for finding the unknown functions in the asymptotic representation of $\mathbf{U}_n$ remains the same. The Hamiltonians corresponding to the bending motion of the shell have the form

$$H_n^{(1)}(p,q)=\left\{\Lambda(p,q)+\left[g_1(q)p^4+m^2f^{-2}(q)g_2(q)p^2+m^4f^{-4}(q)g_3(q)\right]\Lambda^{-1}(p,q)+g_4(q)\right\}^{1/2}$$
$$H_n^{(2)}(p,q)=-H_n^{(1)}(p,q)$$
$$\Lambda(p,q)=\left[p^2+m^2f^{-2}(q)\right]^2,\quad g_1(q)=-(k_1+\nu k_2)^2$$
$$g_2(q)=(1-\nu)^{-1}\left[(2+\nu)(k_1+\nu k_2)(k_2-\nu k_1)-2(k_2+\nu k_1)^2-2(k_1+\nu k_2)^2\right] \tag{3.2}$$
$$g_3(q)=-(1-\nu)^{-1}(k_2+\nu k_1)^2,\quad g_4(q)=\left(k_1^2+2\nu k_1k_2+k_2^2\right)$$

The matrix elements of $\mathbf{L}_{n,0}$ and $\mathbf{N}_n$ will not be given because of their complexity.

4. SOLUTION OF PROBLEM (1.1), (1.2)

Consider the vector-valued function

$$\mathbf{U}^a=\left(\mathbf{E}_{\tau\varepsilon}\left(\mathbf{U}_\tau^\Sigma\right)^T\right)^T+\left(\mathbf{E}_{n\varepsilon}\left(\mathbf{U}_n^\Sigma\right)^T\right)^T \tag{4.1}$$

where

$$U^{\Sigma}_{\tau} = \sum_{l=1}^{4} \sum_{k=0}^{\infty} \varepsilon^{k/2} U^{(l)}_{\tau,k} \exp\left\{ i\varepsilon^{-1} S^{(l)}_{\tau} \right\}$$

$$S^{(l)}_{\tau} = \int_{0}^{t} \omega^{(l)}_{\tau}(t')dt' + \varepsilon^{1/2} p^{(l)}_{\tau}(t)\xi^{(l)}_{\tau} + \frac{1}{2}\varepsilon b^{(l)}_{\tau}(t)\left(\xi^{(l)}_{\tau}\right)^{2}$$

$$\xi^{(l)}_{\tau} = \varepsilon^{-1/2}\left[x - q^{(l)}_{\tau}(t)\right], \quad U^{(l)}_{\tau,k} = \left(u^{(l)}_{\tau,k}, v^{(l)}_{\tau,k}, w^{(l)}_{\tau,k}\right)$$

and the vector-valued function U^{Σ}_{n} is defined by the same formulae with t replaced by $t_1 = \varepsilon t$, and τ and l replaced by n and r ($r = 1, 2$), respectively. Here l and r denote functions corresponding to the Hamiltonians $H^{(l)}_{\tau}$ and $H^{(r)}_{n}$, respectively. For example, $p_{\tau}^{(1)}$, $q_{\tau}^{(1)}$ is a solution of system (2.9) for $H_{\tau} \equiv H^{(1)}_{\tau}$.

By the above constructions U^{α} is a formal asymptotic solution of (1.1). The components of $U^{(l)}_{\tau,k}$ and $U^{(r)}_{n,k}$ are polynomials $\xi^{(l)}_{\tau}$ and $\xi^{(r)}_{n}$ of degree $\sigma^{(l)}_{\tau,k}$ and $\sigma^{(r)}_{n,k}$, respectively, with coefficients containing arbitrary integration constants. We denote by $c^{(l)}_{ki}$, $d^{(r)}_{kj}$ ($i = 0, 1, \ldots, \sigma^{(l)}_{\tau,k}, j = 0, 1, \ldots, \sigma^{(r)}_{n,k}$) the constants appearing in the construction of $P^{(l)}_{\tau,k}$ and $P^{(r)}_{n,k}$, respectively. To find them we substitute (3.1) into the initial conditions (1.2) and equate the coefficients of the same powers of $\varepsilon^{1/2}$. We also use the fact that $\xi^{(l)}_{\tau}|_{t=0} = \xi_{n}|^{(r)}_{t=0} = \zeta$ for any l and r. As a result, we obtain a sequence of equations

$$\begin{gathered}
\sum_{l=1}^{4} u^{(l)}_{\tau,k}\Big|_{t=0} = F^{+}_{1k}(\zeta), \quad \sum_{l=1}^{4} v^{(l)}_{\tau,k}\Big|_{t=0} = F^{+}_{2k}(\zeta) \\
\sum_{l=1}^{4} \delta_{l} u^{(l)}_{\tau,k}\Big|_{t=0} = F^{-}_{1k}(\zeta), \quad \sum_{l=1}^{4} \delta_{l} v^{(l)}_{\tau,k}\Big|_{t=0} = F^{-}_{2k}(\zeta) \\
\left(w^{(1)}_{n,k} \pm w^{(2)}_{n,k}\right)\Big|_{t=0} = F^{\pm}_{3k}(\zeta), \quad k = 0, 1, \ldots
\end{gathered} \tag{4.2}$$

where, in particular

$$F^{+}_{10} = \lambda_{10}(\zeta), \quad F^{+}_{20} = \lambda_{20}(\zeta), \quad F^{+}_{30} = \lambda_{30}(\zeta)$$

$$F^{-}_{10} = -\frac{\eta_{10}(\zeta)}{H^{0}_{\tau}}, \quad F^{-}_{20} = -\frac{\eta_{20}(\zeta)}{H^{0}_{\tau}}$$

$$F^{-}_{30} = -\frac{\eta_{30}(\zeta)}{H^{0}_{n}} - \frac{H^{0}_{\tau}}{H^{0}_{n}}\left[w^{(2)}_{\tau,0} - w^{(1)}_{\tau,0} + \delta_{3}\left(w^{(4)}_{\tau,0} - w^{(3)}_{\tau,0}\right)\right]\Big|_{t=0}$$

and $H^{0}_{\tau} = H^{(1)}_{\tau}(a_0, 0)$ and $H_{n}^{0} = H^{(1)}_{n}(a_0, 0)$. Let $k = 0$. In (4.2) we equate the coefficients of like powers of ζ. Then from the first four equations in (4.2) we find that $\sigma^{(l)}_{\tau,0} = \max\{M_{10}, M_{20}, K_{10}, K_{20}\} = \sigma^{*}_{\tau,0}$ and arrive at a system of $4(\sigma^{*}_{\tau,0} + 1)$ algebraic equations for $c^{(l)}_{0i}$ ($l = 1, \ldots, 4$; $i = 0, 1, \ldots, \sigma^{*}_{\tau,0}$). The last two equations in (4.2) give $\sigma^{(r)}_{n,0} = \max\{M_{30}, K_{30}, \sigma^{*}_{\tau,0}\} = \sigma^{*}_{n,0}$ and a system of $2(\sigma^{*}_{n,0} + 1)$ equations for $d^{(r)}_{0j}$ ($r = 1, 2; j = 0, 1, \ldots, \sigma^{*}_{n,0}$). In a similar way, considering (4.2) for $k \geqslant 1$, one can find the coefficients contained in $P^{(l)}_{\tau,k}$ and $P^{(r)}_{n,k}$.

5. ANALYSIS OF THE SOLUTION

It can be shown [1] that $\operatorname{Im} b^{(l)}_{\tau} > 0$ and $\operatorname{Im} b^{(r)}_{n} > 0$ for any $t \geqslant 0$. It follows that for $t > 0$ the solution (4.1) is the superposition of two packets of flexural waves with centres on the parallel lines $x = q^{(r)}_{n}$ ($r = 1, 2$) and (if $m \neq 0$) four packets of longitudinal waves and four packets of torsional waves with centres $x = q_{\tau}^{(l)}$ ($l = 1, 2, 3, 4$). Then the first and second packets of longitudinal waves corresponding to the Hamiltonians $H_{\tau}^{(1)}$ and $H_{\tau}^{(2)}$ move "bundled" with the first and second packets of torsional waves generated by them, while the third and fourth packets of torsional waves with Hamiltonians $H_{\tau}^{(3)}$ and $H_{\tau}^{(4)}$ generate, respectively, two packets of longitudinal waves.

For $m = 0$ (axially symmetric motion) the solution is unsuitable (see (2.7)). However, in this case the equations of tangential motion (2.1) split into a system describing longitudinal waves and an equation for torsional waves [4]. The solution of the latter can be constructed by the above method. In the axially symmetric case the motion of the shell is the superposition of two packets of longitudinal waves with

Hamiltonians $H_\tau^{(l)} = \delta_l p$ $(l = 1, 2)$, two packets of torsional waves, independent of the longitudinal ones, corresponding to the functions $H_\tau^{(l)} = \delta_l p$ $(l = 3, 4)$, and two packets of flexural waves, for which

$$H_n^{(1)} = \left[\left(1-\nu^2\right)k_2^2(q)+p^4\right]^{1/2}, \quad H_n^{(2)} = -H_n^{(1)} \tag{5.1}$$

Note that if $p_n^{(r)}$ and $p_n^{(l)}$ are close to zero (see (2.7)), the assumption that u_τ, v_τ, w_τ, u_n, v_n, w_n are of order one as $\varepsilon \to 0$ is violated. Thus, for $a_0 > 0$ the vector-valued function (4.1) should be regarded as a formal asymptotic solution of problem (1.1), (1.2) in some interval $t \in [0, T_s]$, where $p_n^{(r)}(t), p_\tau^{(l)}(t) > 0, p_n^{(r)}(t), p_\tau^{(l)}(t) \sim 1$. The quantity T_s depends on the geometry of the shell and the relationships between the parameters appearing in the problem. For example, let $f(x) < 0, f_{\inf} = \inf f(x)$ and let

$$f_{\inf}^2 < m^2 / (a_0^2 + m^2) \tag{5.2}$$

Then, as can be shown by analysing the Hamilton system, to satisfy at least the inequalities $p^{(l)}{}_\tau(t) > 0$ we must take

$$T_s < H_\tau^0 \int_0^{q_*} \left[\left(H_\tau^0\right)^2 - \frac{m^2}{f^2(q)}\right]^{-1/2} dq$$

where q_* can be found from the equation

$$f^2(q_*) = m^2 / (a_0^2 + m^2)$$

If (5.2) is violated, then $p_\tau^{(l)}(t) > 0$ in any interval $[0, T_s]$.

6. EXAMPLE

Consider a conical shell for which $f(x) = 1 - x/\kappa$. Here $\kappa = \operatorname{cosec} \theta$, $\infty < x < \kappa$, where 2θ is the angle at the vertex of the cone. Let $\operatorname{Im} b_0 \geqslant 1$ and let θ be a sufficiently small number. Then initial conditions (1.2) at the vertex of the cone can be neglected. Computations were carried out for axially symmetric packets of flexural waves for $a_0 = 1$, $b_0 = i$, $\kappa = 4.8$, $\lambda_{30} = 1$ with the remaining parameters equal to zero. In Fig. 1 we show the solutions of the Hamilton system, namely, $q_n^{(r)}$ (the solid lines) and $p_n^{(r)}$ (the dashed lines). It can be seen that wave variability decreases in a packet moving towards the vertex of the cone and, conversely, increases in a packet moving in the opposite direction. The graphs of $J^{(r)} = \operatorname{Im} b_n^{(r)}(t_1)$ (the solid lines) and $W^{(r)} = \max | w_n^{(r)}(t_1) |$ (the dashed lines) shown in Fig. 2 indicate that the packet moving towards the vertex of the cone "spreads" more slowly. The wave pattern on the surface of a shell with parameters $h/R = 4 \times 10^{-3}$; $\nu = 0.3$; $E = 6.24 \times 10^{-7}$ kg/(cm s^2), and $\rho = 1.18 \times 10^{-3}$ kg/cm^3 is shown in Fig. 3.

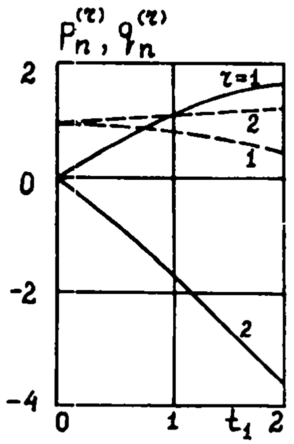

Fig. 1.

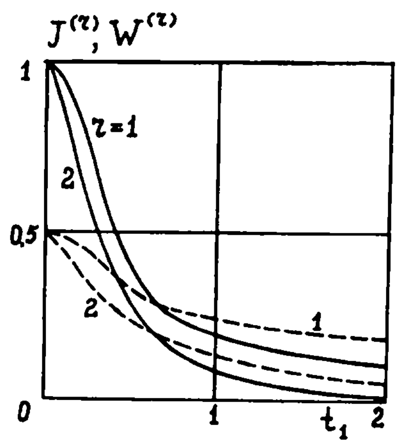

Fig. 2.

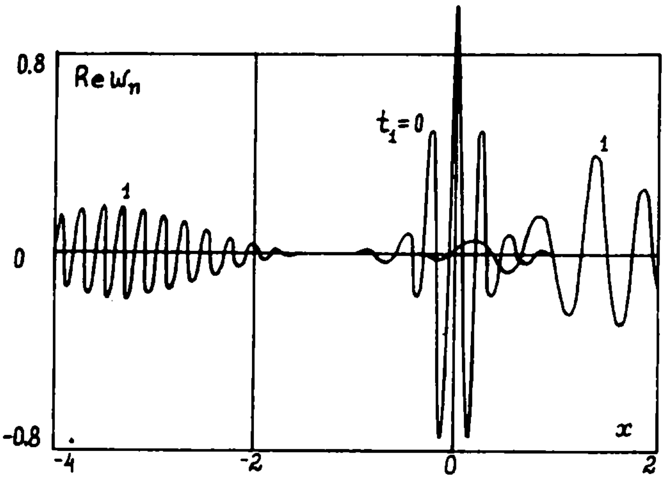

Fig. 3.

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